

Linear response and moderate deviations: hierarchical approach. I

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Abstract

The Moderate Deviations Principle (MDP) is well-understood for sums of independent random variables, worse understood for stationary random sequences, and scantily understood for random fields. Here it is established for a new class of random processes. The approach is promising also for random fields.

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1 Definition, and main result formulated

We examine a class of stationary processes $X = (X_t)_{t \in \mathbb{R}}$, but we are interested only in integrals $\int_{\alpha}^{\beta} X_t dt$ rather than “individual” random variables X_t . Continuity of sample functions is irrelevant as long as these integrals are well-defined. That is, we merely deal with a two-parameter family of random variables, denoted (if only for convenience) by $(\int_{\alpha}^{\beta} X_t dt)_{\alpha < \beta}$ and satisfying

$$(1.1) \quad \int_{\alpha}^{\beta} X_t dt + \int_{\beta}^{\gamma} X_t dt = \int_{\alpha}^{\gamma} X_t dt \quad \text{for } -\infty < \alpha < \beta < \gamma < \infty.$$

Stationarity means measure preserving time shifts that send $\int_{\alpha}^{\beta} X_t dt$ to $\int_{\alpha+s}^{\beta+s} X_t dt$. Thus, the distribution of $\int_{\alpha}^{\beta} X_t dt$ depends on $\beta - \alpha$ only, and we require it to depend measurably:

$$(1.2) \quad \text{the distribution of } \int_0^r X_t dt \text{ is a measurable function of } r;$$

that is, the function $r \mapsto \mathbb{E} \varphi(\int_0^r X_t dt)$ is measurable for every bounded continuous $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ (or equivalently, every bounded Borel measurable φ ; or just $\varphi = \mathbb{1}_{(-\infty, s]}$ for all $s \in \mathbb{R}$; etc). We say that X is *centered*, if

$$(1.3) \quad \mathbb{E} \left| \int_{\alpha}^{\beta} X_t dt \right| < \infty \quad \text{and} \quad \mathbb{E} \int_{\alpha}^{\beta} X_t dt = 0 \quad \text{whenever } \alpha < \beta.$$

We are interested first of all in correlated processes X with continuous sample paths $t \mapsto X_t$. However, our general framework admits uncorrelated processes such as the white noise and the centered Poisson point process, even though their “sample paths” cannot be interpreted as (usual) functions. For the white noise X the random variable $\int_{\alpha}^{\beta} X_t dt$ has the normal distribution $N(0, \beta - \alpha)$. For the centered Poisson point process X the random variable $(\beta - \alpha) + \int_{\alpha}^{\beta} X_t dt$ has the Poisson distribution $P(\beta - \alpha)$.

Our idea of “not too much correlated” process is formalized in the following definition; there, all the four processes (X, X^0, X^-, X^+) are interpreted as above. Independence of processes is independence of the generated σ -fields; and the σ -field generated by X is (by definition) the σ -field generated by random variables $\int_{\alpha}^{\beta} X_t dt$. Two processes X and Y are called identically distributed, if the random vectors $(\int_{\alpha_1}^{\beta_1} X_t dt, \dots, \int_{\alpha_n}^{\beta_n} X_t dt)$ and $(\int_{\alpha_1}^{\beta_1} Y_t dt, \dots, \int_{\alpha_n}^{\beta_n} Y_t dt)$ are identically distributed whenever $\alpha_1 < \beta_1, \dots, \alpha_n < \beta_n$.

1.4 Definition. A centered stationary random process X satisfying (1.2) is *splittable*, if there exist $r > 0$ and $\varepsilon > 0$ such that $\mathbb{E} \exp \varepsilon |\int_0^r X_t dt| < \infty$,¹ and there exists (on some probability space) a triple of random processes X^0, X^-, X^+ such that

- (a) the two processes X^-, X^+ are independent;
- (b) the four processes X, X^0, X^-, X^+ are identically distributed;
- (c) there exists a number $c > 0$ such that for all $a, b > 0$,

$$\mathbb{E} \exp \left(c \left| \int_{-a}^0 X_t^- dt - \int_{-a}^0 X_t^0 dt \right| + c \left| \int_0^b X_t^+ dt - \int_0^b X_t^0 dt \right| \right) \leq 2.$$

¹See also Proposition 2d2 and Remark 2d3.

1.5 Remark. The class of splittable processes is invariant under rescaling on both axes (t and x), that is, under the transition from X to Y where $Y_t = aX_{bt}$ for given parameters $a, b \in (0, \infty)$ (interpreted as $\int_\alpha^\beta Y_t dt = \frac{a}{b} \int_{b\alpha}^{b\beta} X_t dt$, of course). The same holds for $a, b \in \mathbb{R} \setminus \{0\}$ (interpreted as $\int_\alpha^\beta Y_t dt = -\frac{a}{b} \int_{b\beta}^{b\alpha} X_t dt$, if $b < 0$).

1.6 Theorem (“*linear response*”). The following limit exists for every splittable random process X :

$$\lim_{\substack{r \rightarrow \infty, \lambda \rightarrow 0 \\ \lambda \log r \rightarrow 0}} \frac{1}{r\lambda^2} \log \mathbb{E} \exp \lambda \int_0^r X_t dt.$$

That is, for every ε there exist R and δ such that the given expression is ε -close to the limit for all $r \geq R$ and all $\lambda \neq 0$ such that $|\lambda| \log r \leq \delta$.

We denote this limit by $\sigma^2/2$, $\sigma \in [0, \infty)$.

1.7 Corollary (*moderate deviations*). Let X and σ be as above, and $\sigma \neq 0$. Then

$$\lim_{\substack{r \rightarrow \infty, c \rightarrow \infty \\ (c \log r)^2 / r \rightarrow 0}} \frac{1}{c^2} \log \mathbb{P} \left(\int_0^r X_t dt \geq c\sigma\sqrt{r} \right) = -\frac{1}{2}.$$

Unfortunately, the region of moderate deviations ($r \rightarrow \infty$, $c \rightarrow \infty$, $\frac{c^2}{r} \rightarrow 0$) is not covered. The condition $\frac{(c \log r)^2}{r} \rightarrow 0$ leaves a small gap between Corollary 1.7 and large deviations ($\frac{c^2}{r} = \text{const}$).

1.8 Corollary. The distribution of $r^{-1/2} \int_0^r X_t dt$ converges (as $r \rightarrow \infty$) to the normal distribution $N(0, \sigma^2)$.

2 A chain of Hölder inequalities

2a From a splittable process to cumulant generating functions

2a1 Assumption. We restrict ourselves to splittable processes X that satisfy Def. 1.4 with $c = 1$. (This can be ensured, multiplying a given splittable process by a small positive number).

2a2 Remark. Assumption 2a1 is invariant under the transition from $(X_t)_t$ to $(Y_t)_t = (aX_{bt})_t$ provided that $|a| = |b|$.

We consider random variables

$$(2a3) \quad S_r = \frac{1}{\sqrt{r}} \int_0^r X_t dt \quad \text{for } r \in (0, \infty),$$

and their cumulant generating functions

$$(2a4) \quad f_r(\lambda) = \log \mathbb{E} \exp \lambda S_r.$$

Note that $f_r(\lambda) \geq 0$, since $\mathbb{E} \exp \lambda S_r \geq \mathbb{E} (1 + \lambda S_r) = 1$.

2a5 Remark. If $Y_t = aX_{at}$, then $S_r^{(Y)} = \sqrt{a}S_{ar}^{(X)}$ and $f_r^{(Y)}(\lambda) = f_{ar}^{(X)}(\lambda\sqrt{a})$.

2a6 Example. (a) If X is the white noise, then $f_r(\lambda) = \frac{1}{2}\lambda^2$. Also, in this case $(aX_{at})_t$ is distributed like $(\sqrt{a}X_t)_t$.

(b) If X is the centered Poisson point process, then $f_r(\lambda) = (e^{\lambda/\sqrt{r}} - \frac{\lambda}{\sqrt{r}} - 1)r$. Note that $f_r(\lambda) \rightarrow \frac{1}{2}\lambda^2$ as $r \rightarrow \infty$.

2a7 Lemma. For every $r \in (0, \infty)$ there exist random variables U, V, W, Z (on some probability space) such that

- U, V are independent;
- S_r, U, V are identically distributed;
- S_{2r} and W are identically distributed;
- $\sqrt{2r}W = \sqrt{r}U + \sqrt{r}V + Z$;
- $\mathbb{E} \exp |Z| \leq 2$.

Proof. We take processes X^0, X^-, X^+ as in Def. 1.4 and let

$$U = \frac{1}{\sqrt{r}} \int_{-r}^0 X_t^- dt, \quad V = \frac{1}{\sqrt{r}} \int_0^r X_t^+ dt, \quad W = \frac{1}{\sqrt{2r}} \int_{-r}^r X_t^0 dt$$

and $Z = \sqrt{2r}W - \sqrt{r}U - \sqrt{r}V$, then $|Z| \leq \int_{-r}^0 |X_t^- - X_t^0| dt + \int_0^r |X_t^+ - X_t^0| dt$, thus, $\mathbb{E} \exp |Z| \leq \mathbb{E} \exp(\int_{-\infty}^0 |X_t^- - X_t^0| dt + \int_0^\infty |X_t^+ - X_t^0| dt) \leq 2$. \square

Here is a general fact on cumulant generating functions.

2a8 Lemma. If a random variable Z satisfies $\mathbb{E} \exp |Z| \leq 2$ and $\mathbb{E} Z = 0$, then

$$\log \mathbb{E} \exp \lambda Z \leq \lambda^2 \quad \text{for all } \lambda \in [-1, 1].$$

Proof. It is sufficient to prove that $\mathbb{E} (e^{\lambda Z} - 1 - \lambda Z) \leq \lambda^2 (\mathbb{E} e^{|Z|} - 1)$; to this end we'll prove that $e^{\lambda z} - 1 - \lambda z \leq \lambda^2 (e^{|z|} - 1)$ for all $z \in \mathbb{R}$ and $\lambda \in [-1, 1]$. WLOG, $\lambda \in [0, 1]$ (otherwise, use $(-\lambda)$ and $(-z)$).

For $z \geq 0$ the function $\lambda \mapsto (e^{\lambda z} - 1 - \lambda z)/\lambda^2 = \frac{z^2}{2!} + \frac{z^3}{3!}\lambda + \dots$ is increasing on $(0, 1]$, thus, $(e^{\lambda z} - 1 - \lambda z)/\lambda^2 \leq e^z - 1 - z$.

For $z \leq 0$ we have $(e^{\lambda z} - 1 - \lambda z)/\lambda^2 \leq z^2/2$, since $e^{\lambda z} - 1 - \lambda z - \frac{1}{2}(\lambda z)^2 = \frac{1}{6}e^{\theta\lambda z}(\lambda z)^3 \leq 0$ for some $\theta \in [0, 1]$.

Finally, for $z \geq 0$ we have $e^z - 1 - z \leq e^{|z|} - 1$, and for $z \leq 0$ we have $z^2/2 \leq ez^2/2 \leq e^{|z|} - 1$, since $\int_0^{|z|} et \, dt \leq \int_0^{|z|} e^t \, dt$; indeed, $e^t - et = e(e^{t-1} - 1 - (t-1)) \geq 0$. \square

2a9 Proposition. For all $r \in (0, \infty)$ and $p \in (1, \infty)$

$$\begin{aligned} \text{(a)} \quad f_{2r}(\lambda) &\leq \frac{2}{p} f_r\left(\frac{p\lambda}{\sqrt{2}}\right) + \frac{p}{p-1} \cdot \frac{\lambda^2}{2r} && \text{for } |\lambda| \leq \frac{p-1}{p} \sqrt{2r}; \\ \text{(b)} \quad f_{2r}(\lambda) &\geq 2p f_r\left(\frac{\lambda}{p\sqrt{2}}\right) - \frac{1}{p-1} \cdot \frac{\lambda^2}{2r} && \text{for } |\lambda| \leq (p-1)\sqrt{2r}. \end{aligned}$$

Proof. Lemma 2a7 gives U, V, W, Z . By Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left(\exp \frac{\lambda(U+V)}{\sqrt{2}} \cdot \exp \frac{\lambda Z}{\sqrt{2r}} \right) &\leq \\ &\leq \left(\mathbb{E} \exp \frac{p\lambda(U+V)}{\sqrt{2}} \right)^{1/p} \left(\mathbb{E} \exp \frac{p}{p-1} \frac{\lambda Z}{\sqrt{2r}} \right)^{(p-1)/p}. \end{aligned}$$

We note that

$$\begin{aligned} \mathbb{E} \exp \frac{p\lambda(U+V)}{\sqrt{2}} &= \left(\mathbb{E} \exp \frac{p\lambda U}{\sqrt{2}} \right) \left(\mathbb{E} \exp \frac{p\lambda V}{\sqrt{2}} \right) = \\ &= \left(\mathbb{E} \exp \frac{p\lambda S_r}{\sqrt{2}} \right)^2 = \exp 2f_r\left(\frac{p\lambda}{\sqrt{2}}\right), \end{aligned}$$

$$\log \mathbb{E} \exp \frac{p}{p-1} \frac{\lambda Z}{\sqrt{2r}} \leq \left(\frac{p}{p-1} \right)^2 \frac{\lambda^2}{2r} \quad \text{for } |\lambda| \leq \frac{p-1}{p} \sqrt{2r}$$

(by Lemma 2a8), and get (a):

$$\begin{aligned} f_{2r}(\lambda) &= \log \mathbb{E} \exp \lambda S_{2r} = \log \mathbb{E} \exp \lambda W = \\ &= \log \mathbb{E} \exp \left(\frac{\lambda(U+V)}{\sqrt{2}} + \frac{\lambda Z}{\sqrt{2r}} \right) \leq \frac{1}{p} \cdot 2f_r\left(\frac{p\lambda}{\sqrt{2}}\right) + \frac{p-1}{p} \left(\frac{p}{p-1} \right)^2 \frac{\lambda^2}{2r}. \end{aligned}$$

For (b) the argument is similar:

$$\begin{aligned} \mathbb{E} \left(\exp \frac{\lambda W}{p} \cdot \exp \frac{-\lambda Z}{p\sqrt{2r}} \right) &\leq \left(\mathbb{E} \exp \frac{p\lambda W}{p} \right)^{1/p} \left(\mathbb{E} \exp \frac{-p\lambda Z}{(p-1)p\sqrt{2r}} \right)^{(p-1)/p}; \\ \underbrace{\log \mathbb{E} \exp \frac{\lambda(U+V)}{p\sqrt{2}}}_{2f_r(\frac{\lambda}{p\sqrt{2}})} &\leq \frac{1}{p} \underbrace{\log \mathbb{E} \exp \lambda W}_{f_{2r}(\lambda)} + \frac{p-1}{p} \underbrace{\log \mathbb{E} \exp \frac{-\lambda Z}{(p-1)\sqrt{2r}}}_{\leq \frac{\lambda^2}{(p-1)^2 \cdot 2r}}. \end{aligned}$$

\square

2a10 Remark. More generally, for all $r, s \in (0, \infty)$ and $p \in (1, \infty)$,

$$\begin{aligned}
(a) \quad f_{r+s}(\lambda) &\leq \frac{1}{p} f_r \left(p \lambda \sqrt{\frac{r}{r+s}} \right) + \frac{1}{p} f_s \left(p \lambda \sqrt{\frac{s}{r+s}} \right) + \frac{p}{p-1} \cdot \frac{\lambda^2}{r+s} \\
&\quad \text{for } |\lambda| \leq \frac{p-1}{p} \sqrt{r+s}; \\
(b) \quad f_{r+s}(\lambda) &\geq p f_r \left(\frac{\lambda}{p} \sqrt{\frac{r}{r+s}} \right) + p f_s \left(\frac{\lambda}{p} \sqrt{\frac{s}{r+s}} \right) - \frac{1}{p-1} \cdot \frac{\lambda^2}{r+s} \\
&\quad \text{for } |\lambda| \leq (p-1) \sqrt{r+s}.
\end{aligned}$$

To this end, take $U = \frac{1}{\sqrt{r}} \int_{-r}^0 X_t^- dt$, $V = \frac{1}{\sqrt{s}} \int_0^s X_t^+ dt$, $W = \frac{1}{\sqrt{r+s}} \int_{-r}^s X_t^0 dt$ in the proof of 2a7.

2b Upper bounds

In this subsection we investigate an arbitrary family of functions $f_r : \mathbb{R} \rightarrow [0, \infty]$ for $r \in (0, \infty)$ such that

$$(2b1) \quad f_{2r}(\lambda) \leq \frac{2}{p} f_r \left(\frac{p\lambda}{\sqrt{2}} \right) + \frac{p}{p-1} \cdot \frac{\lambda^2}{2r}$$

whenever $0 < r < \infty$, $1 < p < \infty$ and $\frac{|\lambda|}{\sqrt{2r}} \leq \frac{p-1}{p}$. (The functions (2a4) satisfy (2b1) by Prop. 2a9(a).)

If a family $(f_r)_r$ satisfies (2b1), then for arbitrary $s \in (0, \infty)$ the rescaled family $(g_r)_r$ defined by

$$(2b2) \quad g_r(\lambda) = f_{s^2 r}(s\lambda)$$

satisfies (2b1) (which is evidently related to Remark 2a5).

2b3 Lemma. Let $a \geq 1$, $\varepsilon \geq 0$, $r > 0$, and $\frac{\varepsilon}{\sqrt{r}} \leq \sqrt{2} - 1$. If

$$f_r(\varepsilon\lambda) \leq (a-1)\lambda^2 \quad \text{for } |\lambda| \leq 1,$$

then

$$f_{2r}(\varepsilon\lambda) \leq \left(a \left(1 + \frac{\varepsilon}{\sqrt{r}} \right) - 1 \right) \lambda^2 \quad \text{for } |\lambda| \leq 1.$$

2b4 Remark. If this lemma holds for ε and r , then for arbitrary $s \in (0, \infty)$ it holds also for $s\varepsilon$ and $s^2 r$ due to the rescaling (2b2). All relevant functions of ε, r depend only on the invariant combination ε/\sqrt{r} . (Also a and λ are invariant.) Therefore it is sufficient to prove Lemma 2b3 for $r = 1$ only. (This argument will be used many times.)

Proof of Lemma 2b3. We restrict ourselves to the case $r = 1$ according to Remark 2b4. Assuming $\varepsilon \neq 0$ we take $p = 1 + \varepsilon$, note that $p \leq \sqrt{2}$, $\frac{p-1}{p} \geq \frac{\varepsilon}{\sqrt{2}}$, and apply (2b1) to $\varepsilon\lambda$ in place of λ , getting two summands. The second summand is $\frac{p}{p-1} \frac{\varepsilon^2 \lambda^2}{2} \leq \frac{\varepsilon \lambda^2}{\sqrt{2}} \leq \varepsilon \lambda^2$. The first summand does not exceed $\frac{2}{1+\varepsilon} (a-1)^{\frac{1}{2}} (1+\varepsilon)^2 \lambda^2 \leq (1+\varepsilon)(a-1) \lambda^2$. \square

Iterating the transition $r \mapsto 2r$ we multiply a by $(1 + \frac{\varepsilon}{\sqrt{r}})(1 + \frac{\varepsilon}{\sqrt{2r}})(1 + \frac{\varepsilon}{\sqrt{4r}}) \cdots \leq \exp(\frac{\sqrt{2}}{\sqrt{2}-1} \frac{\varepsilon}{\sqrt{r}})$ and get the following.

2b5 Proposition. Let $a \geq 1$, $\varepsilon \geq 0$, $r > 0$, and $\frac{\varepsilon}{\sqrt{r}} \leq \sqrt{2} - 1$. If

$$f_r(\varepsilon\lambda) \leq (a-1)\lambda^2 \quad \text{for } |\lambda| \leq 1,$$

then, for every $n = 0, 1, 2, \dots$,

$$f_{2^n r}(\varepsilon\lambda) \leq \left(a \exp\left(\frac{\sqrt{2}}{\sqrt{2}-1} \frac{\varepsilon}{\sqrt{r}}\right) - 1 \right) \lambda^2 \quad \text{for } |\lambda| \leq 1.$$

2b6 Lemma. Let $a, b, c, \delta \geq 0$, $b\delta < 1$, and $r > 0$. If

$$f_r(\lambda) \leq \frac{a\lambda^2}{1 - \frac{b|\lambda|}{\sqrt{r}}} + \frac{c|\lambda|}{\sqrt{r}} \quad \text{for } |\lambda| \leq \delta\sqrt{r},$$

then

$$f_{2r}(\lambda) \leq \frac{a\lambda^2}{1 - \frac{(b+1)|\lambda|}{\sqrt{2r}}} + \frac{(2c+1)|\lambda|}{\sqrt{2r}} \quad \text{for } |\lambda| \leq \frac{\delta}{1+\delta}\sqrt{2r}.$$

Proof. We restrict ourselves to the case $r = 1$ according to Remark 2b4.² Assuming $\lambda \neq 0$ we take

$$p = \frac{1}{1 - \frac{|\lambda|}{\sqrt{2}}},$$

note that

- $1 - \frac{bp|\lambda|}{\sqrt{2}} = p(1 - \frac{(b+1)|\lambda|}{\sqrt{2}})$ (since $1 = p - p\frac{|\lambda|}{\sqrt{2}}$);
- $\left| \frac{p\lambda}{\sqrt{2}} \right| = \frac{\frac{|\lambda|}{\sqrt{2}}}{1 - \frac{|\lambda|}{\sqrt{2}}} \leq \delta$;
- $\frac{p-1}{p} = \frac{|\lambda|}{\sqrt{2}}$;

²Invariant are $b, c, \delta, \lambda^2/r, a\lambda^2$.

and apply (2b1), getting two summands. The second summand is $\frac{p}{p-1} \frac{\lambda^2}{2} = \frac{|\lambda|}{\sqrt{2}}$. The first summand is

$$\begin{aligned} \frac{2}{p} f_1\left(\frac{p\lambda}{\sqrt{2}}\right) &\leq \frac{2}{p} \left(\frac{a\left(\frac{p\lambda}{\sqrt{2}}\right)^2}{1 - b\left|\frac{p\lambda}{\sqrt{2}}\right|} + c\left|\frac{p\lambda}{\sqrt{2}}\right| \right) = \\ &= \frac{ap\lambda^2}{1 - \frac{bp|\lambda|}{\sqrt{2}}} + \frac{2c|\lambda|}{\sqrt{2}} = \frac{a\lambda^2}{1 - \frac{(b+1)|\lambda|}{\sqrt{2}}} + \frac{2c|\lambda|}{\sqrt{2}}. \quad \square \end{aligned}$$

2b7 Proposition. Let $a, \delta \geq 0$, and $r > 0$. If

$$f_r(\lambda) \leq a\lambda^2 \quad \text{for } |\lambda| \leq \delta\sqrt{r},$$

then (for every $n = 0, 1, 2, \dots$)

$$f_{2^n r}(\lambda) \leq \frac{a\lambda^2}{1 - \frac{n|\lambda|}{2^{n/2}\sqrt{r}}} + \frac{2^{n/2}|\lambda|}{\sqrt{r}} \quad \text{for } |\lambda| \leq \frac{\delta}{1 + n\delta} 2^{n/2}\sqrt{r}.$$

Proof. We prove a bit stronger inequality, with the second summand $(1 - 2^{-n}) \frac{2^{n/2}|\lambda|}{\sqrt{r}}$ instead of $\frac{2^{n/2}|\lambda|}{\sqrt{r}}$, by induction in n . Case $n = 0$ is trivial. If the claim holds for n , then Lemma 2b6 applies to $2^n r$, $b = n$, $c = (1 - 2^{-n})2^n = 2^n - 1$, and $\frac{\delta}{1 + n\delta}$, giving

$$f_{2^{n+1}r}(\lambda) \leq \frac{a\lambda^2}{1 - \frac{(n+1)|\lambda|}{\sqrt{2^{n+1}r}}} + \frac{(2^{n+1} - 1)|\lambda|}{\sqrt{2^{n+1}r}} \quad \text{for } |\lambda| \leq \frac{\delta}{1 + (n+1)\delta} \sqrt{2^{n+1}r}. \quad \square$$

2b8 Theorem. Let $\varepsilon \in (0, \sqrt{2} - 1]$ and $r, a \in (0, \infty)$. If

$$f_r(\varepsilon\lambda) \leq a\lambda^2 \quad \text{for } |\lambda| \leq \sqrt{r},$$

then, for every $n = 1, 2, \dots$,

$$f_{2^n r}(\varepsilon\lambda) \leq a\lambda^2 + C\varepsilon \left(a + \frac{1}{r}\right) \frac{1 + V}{1 - \varepsilon V} \lambda^2 \quad \text{for } |\lambda| \leq \frac{2^{n/2}\sqrt{r}}{\varepsilon n + \max(\varepsilon\sqrt{2n}, 1)},$$

where

$$C = \frac{1}{\varepsilon} \left(\exp\left(\frac{\sqrt{2}}{\sqrt{2}-1}\varepsilon\right) - 1 \right), \quad V = \frac{n}{2^{n/2}} \frac{|\lambda|}{\sqrt{r}}.$$

Note that the condition on λ may be rewritten as

$$(2b9) \quad \left(\varepsilon + \max\left(\varepsilon\sqrt{\frac{2}{n}}, \frac{1}{n}\right) \right) V \leq 1;$$

it evidently implies $\varepsilon V < 1$.

Remark 2b4 applies; Theorem 2b8 is scaling invariant.³

We start proving Theorem 2b8. According to Remark 2b4 we restrict ourselves to the case $r = 1$. The following four lemmas are fragments of the proof; they will not be reused later. Throughout we assume that $\varepsilon, a > 0$, $f_1(\varepsilon\lambda) \leq a\lambda^2$ for $|\lambda| \leq 1$, and use $C \geq \frac{\sqrt{2}}{\sqrt{2}-1}$, V such that $\exp(\frac{\sqrt{2}}{\sqrt{2}-1}\varepsilon) = 1 + C\varepsilon$, $|\lambda| = \frac{2^{n/2}V}{n}$, and $\varepsilon V < 1$ (that is, $|\lambda| < \frac{2^{n/2}}{\varepsilon n}$).

2b10 Lemma. Let $\varepsilon \leq \sqrt{2} - 1$, $m \in \{0, 1, 2, \dots\}$, and $|\lambda| \leq 1$. Then

$$f_{2^m}(\varepsilon\lambda) \leq a\lambda^2 + C\varepsilon(a+1)\lambda^2.$$

Proof. Prop. 2b5 with $r = 1$ and $a + 1$ in place of a gives

$$f_{2^m}(\varepsilon\lambda) \leq ((1 + C\varepsilon)(a + 1) - 1)\lambda^2 \quad \text{for } |\lambda| \leq 1.$$

And $(1 + C\varepsilon)(a + 1) - 1 = a + C\varepsilon(a + 1)$. □

2b11 Lemma. Let $\varepsilon \leq \sqrt{2} - 1$, $m \in \{0, 1, 2, \dots, n-1\}$, and $|\lambda| \leq \frac{2^{(n-m)/2}}{1+n2^{-m/2}\varepsilon}$. Then

$$f_{2^n}(\varepsilon\lambda) \leq \frac{a + C\varepsilon(a + 1)}{1 - \varepsilon V} \lambda^2 + \frac{2^{n-m}}{n} \varepsilon V.$$

Proof. By Lemma 2b10, $f_{2^m}(\lambda) \leq \frac{A}{\varepsilon^2} \lambda^2$ for $|\lambda| \leq \varepsilon$, where $A = a + C\varepsilon(a + 1)$. Thus, the conditions of Prop. 2b7 are satisfied for $r = 2^m$, $\delta = 2^{-m/2}\varepsilon$ and $a = A/\varepsilon^2$. Taking also $n - m$ in place of n we get from Prop. 2b7

$$f_{2^{n-m}2^m}(\lambda) \leq \frac{A}{\varepsilon^2} \frac{\lambda^2}{1 - \frac{(n-m)|\lambda|}{2^{(n-m)/2}\sqrt{2^m}}} + \frac{2^{(n-m)/2}|\lambda|}{\sqrt{2^m}}$$

for $|\lambda| \leq \frac{\delta}{1+(n-m)\delta} 2^{(n-m)/2} \sqrt{2^m}$. Therefore,

$$f_{2^n}(\lambda) \leq \frac{A}{\varepsilon^2} \frac{\lambda^2}{1 - \frac{n|\lambda|}{2^{n/2}}} + 2^{\frac{n}{2}-m} |\lambda| \quad \text{for } |\lambda| \leq \frac{\delta}{1+n\delta} 2^{n/2}.$$

That is,

$$f_{2^n}(\varepsilon\lambda) \leq \frac{A}{1 - \frac{n\varepsilon|\lambda|}{2^{n/2}}} \lambda^2 + 2^{\frac{n}{2}-m} \varepsilon |\lambda| = \frac{A}{1 - \varepsilon V} \lambda^2 + \frac{2^{n-m}}{n} \varepsilon V$$

for $|\lambda| \leq \frac{\delta}{1+n\delta} \frac{1}{\varepsilon} 2^{n/2} = \frac{2^{(n-m)/2}}{1+n2^{-m/2}\varepsilon}$. □

³Invariant are ε , λ^2/r , $a\lambda^2$.

Taking into account that

$$a + C\varepsilon \frac{a+1+aV}{1-\varepsilon V} - \frac{a+C\varepsilon(a+1)}{1-\varepsilon V} = \frac{(C-1)\varepsilon aV}{1-\varepsilon V} \geq 0$$

we get the following.

2b12 Corollary. Let $\varepsilon \leq \sqrt{2} - 1$, $m \in \{0, 1, 2, \dots, n-1\}$, and $|\lambda| \leq \frac{2^{(n-m)/2}}{1+n2^{-m/2}\varepsilon}$. Then

$$f_{2^n}(\varepsilon\lambda) \leq a\lambda^2 + C\varepsilon(a+1)\frac{1+V}{1-\varepsilon V}\lambda^2 - C\varepsilon\frac{V}{1-\varepsilon V}\lambda^2 + \frac{2^{n-m}}{n}\varepsilon V.$$

Lemma 2b10 for $m = n$ gives Theorem 2b8 in the case $|\lambda| \leq 1$, that is, $\frac{V}{n} \leq 2^{-n/2}$. For greater $|\lambda|$ (and V) we'll obtain Theorem 2b8 from Corollary 2b12, choosing m as follows. (Recall (2b9).)

2b13 Lemma. If $\frac{V}{n} > 2^{-n/2}$ and $(\varepsilon + \frac{1}{n})V \leq 1$ (that is, $1 < |\lambda| \leq \frac{2^{n/2}}{\varepsilon n+1}$), then there exists (evidently unique) $m \in \{0, 1, 2, \dots, n-1\}$ such that

$$1 \leq \frac{(1-\varepsilon V)n}{2^{m/2}V} < \sqrt{2}.$$

Proof. The greatest $m \in \mathbb{Z}$ such that $2^{m/2} \leq \frac{(1-\varepsilon V)n}{V}$ satisfies $m < n$, since $2^{n/2} > \frac{n}{V} \geq \frac{(1-\varepsilon V)n}{V}$; it also satisfies $m \geq 0$, since $(\varepsilon + \frac{1}{n})V \leq 1 \implies \varepsilon nV + V \leq n \implies 1 \leq \frac{(1-\varepsilon V)n}{V}$. \square

From now on, m is chosen as above. Note that $1 \leq \frac{(1-\varepsilon V)n}{2^{m/2}V} \implies (2^{m/2} + \varepsilon n)V \leq n \implies |\lambda| \leq \frac{2^{(n-m)/2}}{1+n2^{-m/2}\varepsilon}$, thus, Corollary 2b12 applies, and so, the next lemma completes the proof of Theorem 2b8.

2b14 Lemma. Let $1 < |\lambda| \leq \frac{2^{n/2}}{\varepsilon n + \max(\varepsilon\sqrt{2n}, 1)}$. Then

$$\frac{2^{n-m}}{n}\varepsilon V \leq C\varepsilon\frac{V}{1-\varepsilon V}\lambda^2.$$

Proof. We rewrite the given restriction $|\lambda| \leq \frac{2^{n/2}}{\varepsilon n + \varepsilon\sqrt{2n}}$ in terms of V :

$$\left(1 + \sqrt{\frac{2}{n}}\right)\varepsilon V \leq 1.$$

We also eliminate λ from the needed inequality:

$$C \cdot 2^m V^2 \geq n(1 - \varepsilon V).$$

By 2b13, $2 \cdot 2^m V^2 > n^2(1 - \varepsilon V)^2$. Thus, it is sufficient to prove that $Cn^2(1 - \varepsilon V)^2 \geq 2n(1 - \varepsilon V)$, that is, $\varepsilon V \leq 1 - \frac{2}{Cn}$. To this end it is sufficient to prove that $(1 + \sqrt{\frac{2}{n}})(1 - \frac{2}{Cn}) \geq 1$, that is, $\sqrt{2}n - \frac{2\sqrt{2}}{C} \geq \frac{2}{C}\sqrt{n}$, and we may do it for $n = 1$ only: $\sqrt{2} - \frac{2\sqrt{2}}{C} \geq \frac{2}{C}$, that is, $\sqrt{2} \geq \frac{2(\sqrt{2}+1)}{C}$, since $C \geq \frac{2}{2-\sqrt{2}}$. \square

2c Lower bounds

In this subsection we investigate an arbitrary family of functions $f_r : \mathbb{R} \rightarrow [0, \infty]$ for $r \in (0, \infty)$ such that

$$(2c1) \quad f_{2r}(\lambda) \geq 2pf_r\left(\frac{\lambda}{p\sqrt{2}}\right) - \frac{1}{p-1} \cdot \frac{\lambda^2}{2r}$$

whenever $0 < r < \infty$, $1 < p < \infty$ and $\frac{|\lambda|}{\sqrt{2r}} \leq p-1$. (The functions (2a4) satisfy (2c1) by Prop. 2a9(b).)

If a family $(f_r)_r$ satisfies (2c1), then for arbitrary $s \in (0, \infty)$ the rescaled family $(g_r)_r$ defined by (2b2), that is, $g_r(\lambda) = f_{s^2r}(s\lambda)$, satisfies (2c1).

2c2 Lemma. Let $a \geq 1$, $\varepsilon \geq 0$, $r > 0$, and $\frac{\varepsilon}{\sqrt{r}} < \sqrt{2}$. If

$$f_r(\varepsilon\lambda) \geq (a-1)\lambda^2 \quad \text{for } |\lambda| \leq 1,$$

then

$$f_{2r}(\varepsilon\lambda) \geq \left(a\left(1 - \frac{\varepsilon}{\sqrt{2r}}\right) - 1\right)\lambda^2 \quad \text{for } |\lambda| \leq 1.$$

Proof. We restrict ourselves to the case $r = 1$ according to Remark 2b4.⁴ We take

$$p = \frac{1}{1 - \frac{\varepsilon}{\sqrt{2}}},$$

note that $\frac{\varepsilon^2}{2} = \left(\frac{p-1}{p}\right)^2 \leq \frac{(p-1)^2}{p}$, and apply (2c1) to $\varepsilon\lambda$ in place of λ ;

$$f_2(\varepsilon\lambda) \geq 2p(a-1)\frac{\lambda^2}{2p^2} - \frac{1}{p-1} \frac{\varepsilon^2\lambda^2}{2} = \left(\frac{a-1}{p} - \frac{\varepsilon^2}{2(p-1)}\right)\lambda^2 \geq \left(\frac{a}{p} - 1\right)\lambda^2. \quad \square$$

Iterating the transition $r \mapsto 2r$ we multiply a by $\left(1 - \frac{\varepsilon}{\sqrt{2r}}\right)\left(1 - \frac{\varepsilon}{\sqrt{4r}}\right)\left(1 - \frac{\varepsilon}{\sqrt{8r}}\right) \dots$; this product cannot be less than $1 - (\sqrt{2} + 1)\frac{\varepsilon}{\sqrt{r}}$, since $(1 - a\varepsilon)(1 - b\varepsilon) \geq 1 - (a+b)\varepsilon$ for $a, b \geq 0$, and $\frac{\varepsilon}{\sqrt{2r}} + \frac{\varepsilon}{\sqrt{4r}} + \dots = (\sqrt{2} + 1)\frac{\varepsilon}{\sqrt{r}}$. Thus, we get the following.

2c3 Proposition. Let $a \geq 1$, $\varepsilon \geq 0$, $r > 0$, and $\frac{\varepsilon}{\sqrt{r}} < \sqrt{2}$. If

$$f_r(\varepsilon\lambda) \geq (a-1)\lambda^2 \quad \text{for } |\lambda| \leq 1,$$

then, for every $n = 0, 1, 2, \dots$,

$$f_{2^n r}(\varepsilon\lambda) \geq \left(a\left(1 - (\sqrt{2} + 1)\frac{\varepsilon}{\sqrt{r}}\right) - 1\right)\lambda^2 \quad \text{for } |\lambda| \leq 1.$$

⁴Invariant are ε/\sqrt{r} , a , λ .

2c4 Lemma. Let $a, b, c, \delta \geq 0$ and $r > 0$. If

$$f_r(\lambda) \geq \frac{a\lambda^2}{1 + \frac{b|\lambda|}{\sqrt{r}}} - \frac{c|\lambda|}{\sqrt{r}} \quad \text{for } |\lambda| \leq \delta\sqrt{r},$$

then

$$f_{2r}(\lambda) \geq \frac{a\lambda^2}{1 + \frac{(b+1)|\lambda|}{\sqrt{2r}}} - \frac{(2c+1)|\lambda|}{\sqrt{2r}} \quad \text{for } (1-\delta)|\lambda| \leq \delta\sqrt{2r}.$$

(It may be that $\delta \geq 1$, and then λ is not restricted.)

Proof. We restrict ourselves to the case $r = 1$ according to Remark 2b4.⁵ Assuming $\lambda \neq 0$ we take

$$p = 1 + \frac{|\lambda|}{\sqrt{2}},$$

note that $(1-\delta)|\lambda| \leq \delta\sqrt{2} \implies \frac{|\lambda|}{p\sqrt{2}} \leq \delta$ (also for $\delta \geq 1$), and apply (2c1);

$$\begin{aligned} f_2(\lambda) &\geq 2p \left(\frac{\frac{a\lambda^2}{2p^2}}{1 + b\left|\frac{\lambda}{p\sqrt{2}}\right|} - c\left|\frac{\lambda}{p\sqrt{2}}\right| \right) - \frac{1}{p-1} \frac{\lambda^2}{2} = \\ &= \frac{a\lambda^2}{p + \frac{b|\lambda|}{\sqrt{2}}} - \sqrt{2}c|\lambda| - \frac{\lambda^2}{2(p-1)} = \\ &= \frac{a\lambda^2}{1 + \frac{|\lambda|}{\sqrt{2}} + \frac{b|\lambda|}{\sqrt{2}}} - \sqrt{2}c|\lambda| - \frac{|\lambda|}{\sqrt{2}} = \frac{a\lambda^2}{1 + \frac{(b+1)|\lambda|}{\sqrt{2}}} - (2c+1)\frac{|\lambda|}{\sqrt{2}}. \quad \square \end{aligned}$$

2c5 Proposition. Let $a, \delta \geq 0$, and $r > 0$. If

$$f_r(\lambda) \geq a\lambda^2 \quad \text{for } |\lambda| \leq \delta\sqrt{r},$$

then (for every $n = 0, 1, 2, \dots$)

$$f_{2^n r}(\lambda) \geq \frac{a\lambda^2}{1 + \frac{n|\lambda|}{2^{n/2}\sqrt{r}}} - \frac{2^{n/2}|\lambda|}{\sqrt{r}} \quad \text{for } (1-n\delta)|\lambda| \leq \delta 2^{n/2}\sqrt{r}.$$

(It may be that $n\delta \geq 1$, and then λ is not restricted.)

⁵Invariant are $b, c, \delta, \lambda^2/r, a\lambda^2$.

Proof. We prove a bit stronger inequality, with the second summand $-(1 - 2^{-n})\frac{2^{n/2}|\lambda|}{\sqrt{r}}$ instead of $-\frac{2^{n/2}|\lambda|}{\sqrt{r}}$, by induction in n . Case $n = 0$ is trivial. If the claim holds for n , then Lemma 2c4 applies to $2^n r$, $b = n$, $c = (1 - 2^{-n})2^n = 2^n - 1$, and $\frac{\delta}{1-n\delta}$ (interpreted as $+\infty$ if $n\delta \geq 1$), giving

$$f_{2^{n+1}r}(\lambda) \geq \frac{a\lambda^2}{1 + \frac{(n+1)|\lambda|}{\sqrt{2^{n+1}r}}} - \frac{(2^{n+1} - 1)|\lambda|}{\sqrt{2^{n+1}r}}$$

for $(1 - \frac{\delta}{1-n\delta})|\lambda| \leq \frac{\delta}{1-n\delta}\sqrt{2^{n+1}r}$, that is, $(1 - n\delta - \delta)|\lambda| \leq \delta\sqrt{2^{n+1}r}$ (and λ is not restricted if $(n+1)\delta \geq 1$). \square

2c6 Theorem. Let $\varepsilon \in (0, \sqrt{2})$ and $r, a \in (0, \infty)$. If

$$f_r(\varepsilon\lambda) \geq a\lambda^2 \quad \text{for } |\lambda| \leq \sqrt{r},$$

then, for every $n = 1, 2, \dots$,

$$f_{2^n r}(\varepsilon\lambda) \geq a\lambda^2 - (\sqrt{2} + 1)\varepsilon\left(a + \frac{1}{r}\right)(1 + V)\lambda^2 \quad \text{for } |\lambda| \leq 2^{n/2}\sqrt{r},$$

where

$$V = \frac{n}{2^{n/2}} \frac{|\lambda|}{\sqrt{r}}.$$

We start proving Theorem 2c6. According to Remark 2b4 we restrict ourselves to the case $r = 1$.⁶ The following two lemmas are fragments of the proof; they will not be reused later. Throughout we assume that $\varepsilon, a > 0$, $f_1(\varepsilon\lambda) \geq a\lambda^2$ for $|\lambda| \leq 1$, and use V such that $|\lambda| = \frac{2^{n/2}V}{n}$.

2c7 Lemma. Let $\varepsilon < \sqrt{2}$, $m \in \{0, 1, 2, \dots\}$, and $|\lambda| \leq 1$. Then

$$f_{2^m}(\varepsilon\lambda) \geq a\lambda^2 - (\sqrt{2} + 1)\varepsilon(a + 1)\lambda^2.$$

Proof. Prop. 2c3 with $r = 1$ and $a + 1$ in place of a gives

$$f_{2^m}(\varepsilon\lambda) \geq ((1 - (\sqrt{2} + 1)\varepsilon)(a + 1) - 1)\lambda^2 \quad \text{for } |\lambda| \leq 1.$$

And $(1 - (\sqrt{2} + 1)\varepsilon)(a + 1) - 1 = a - (\sqrt{2} + 1)\varepsilon(a + 1)$. \square

2c8 Lemma. Let $\varepsilon < \sqrt{2}$, $m \in \{0, 1, 2, \dots, n-1\}$, and $(1 - (n-m)2^{-m/2}\varepsilon)|\lambda| \leq 2^{(n-m)/2}$. Then

$$f_{2^n}(\varepsilon\lambda) \geq \frac{a - (\sqrt{2} + 1)\varepsilon(a + 1)}{1 + \varepsilon V}\lambda^2 - \frac{2^{n-m}}{n}\varepsilon V.$$

⁶Invariant are ε , λ^2/r , $a\lambda^2$.

Proof. By Lemma 2c7, $f_{2^m}(\lambda) \geq \frac{A}{\varepsilon^2} \lambda^2$ for $|\lambda| \leq \varepsilon$, where $A = a - (\sqrt{2} + 1)\varepsilon(a + 1)$. Thus, the conditions of Prop. 2c5 are satisfied for $r = 2^m$, $\delta = 2^{-m/2}\varepsilon$ and $a = A/\varepsilon^2$. Taking also $n - m$ in place of n we get from Prop. 2c5

$$f_{2^{n-m}2^m}(\lambda) \geq \frac{A}{\varepsilon^2} \frac{\lambda^2}{1 + \frac{(n-m)|\lambda|}{2^{(n-m)/2}\sqrt{2^m}}} - \frac{2^{(n-m)/2}|\lambda|}{\sqrt{2^m}}$$

for $(1 - (n - m)\delta)|\lambda| \leq \delta 2^{(n-m)/2} \sqrt{2^m}$. Therefore,

$$f_{2^n}(\lambda) \geq \frac{A}{\varepsilon^2} \frac{\lambda^2}{1 + \frac{n|\lambda|}{2^{n/2}}} - 2^{\frac{n}{2}-m}|\lambda| \quad \text{for } (1 - (n - m)\delta)|\lambda| \leq \delta 2^{n/2}.$$

That is,

$$f_{2^n}(\varepsilon\lambda) \geq \frac{A}{1 + \frac{n\varepsilon|\lambda|}{2^{n/2}}} \lambda^2 - 2^{\frac{n}{2}-m}\varepsilon|\lambda| = \frac{A}{1 + \varepsilon V} \lambda^2 - \frac{2^{n-m}}{n} \varepsilon V$$

for $(1 - (n - m)2^{-m/2}\varepsilon)|\lambda| \leq 2^{(n-m)/2}$. □

Taking into account that

$$\begin{aligned} \frac{1}{\varepsilon} \left(a - \frac{a - (\sqrt{2} + 1)\varepsilon(a + 1)}{1 + \varepsilon V} \right) &= \frac{(\sqrt{2} + 1)(a + 1) + aV}{1 + \varepsilon V} \leq \\ (\sqrt{2} + 1)(a + 1) + aV &= (\sqrt{2} + 1)(a + 1)(1 + V) - (\sqrt{2}(a + 1) + 1)V \end{aligned}$$

and waiving the factor $1 - (n - m)2^{-m/2}\varepsilon$ we get the following.

2c9 Corollary. Let $\varepsilon < \sqrt{2}$, $m \in \{0, 1, 2, \dots, n - 1\}$, and $|\lambda| \leq 2^{(n-m)/2}$. Then

$$f_{2^n}(\varepsilon\lambda) \geq a\lambda^2 - (\sqrt{2} + 1)\varepsilon(a + 1)(1 + V)\lambda^2 + \varepsilon V(\sqrt{2}(a + 1) + 1)\lambda^2 - \frac{2^{n-m}}{n} \varepsilon V.$$

Now we prove Theorem 2c6 as follows. For $|\lambda| \leq 1$ we just apply Lemma 2c7 with $m = n$. For $1 \leq |\lambda| \leq 2^{n/2}$ we choose $m \in \{0, 1, 2, \dots, n - 1\}$ such that

$$2^{\frac{n-m-1}{2}} \leq |\lambda| \leq 2^{\frac{n-m}{2}},$$

apply Corollary 2c9 and note that $\frac{2^{n-m}}{n} \varepsilon V \leq \varepsilon V(\sqrt{2}(a + 1) + 1)\lambda^2$, since $\frac{2^{n-m}}{n} \leq 2 \cdot 2^{n-m-1} \leq 2\lambda^2 \leq (\sqrt{2} + 1)\lambda^2$.

2d More on the cumulant generating functions

First, a general fact.

2d1 Lemma. Let X be a random variable such that $\mathbb{E} \exp |X| < \infty$ and $\mathbb{E} X = 0$. Then its cumulant generating function

$$f(\lambda) = \log \mathbb{E} \exp \lambda X$$

satisfies

$$\left| f(\lambda) - \frac{1}{2} f''(0) \lambda^2 \right| \leq \frac{41}{6e^3} \left(\frac{|\lambda|}{1-|\lambda|} \right)^3 (\exp f(-1) + \exp f(1)) \quad \text{for } |\lambda| < 1.$$

Proof. In terms of $g(\lambda) = \mathbb{E} \exp \lambda X$ we have $f(\lambda) = \log g(\lambda)$ and

$$f'''(\lambda) = \frac{g'''(\lambda)}{g(\lambda)} - 3 \frac{g'(\lambda)g''(\lambda)}{g^2(\lambda)} + 2 \frac{g'^3(\lambda)}{g^3(\lambda)}.$$

Applying the inequality $u^k e^{-u} \leq \left(\frac{k}{e}\right)^k$ to $u = (1-|\lambda|)|X|$ we get

$$\begin{aligned} |g^{(k)}(\lambda)| &= |\mathbb{E} X^k \exp \lambda X| \leq \mathbb{E} |X|^k \exp |\lambda| |X| = \\ &= \mathbb{E} (|X|^k \exp(-(1-|\lambda|)|X|) \exp |X|) \leq \frac{1}{(1-|\lambda|)^k} \left(\frac{k}{e}\right)^k \mathbb{E} \exp |X|; \end{aligned}$$

also, $g(\lambda) \geq 1$ (since $\exp(\lambda X) \geq 1 + \lambda X$); thus,

$$\begin{aligned} |f'''(\lambda)| &\leq |g'''(\lambda)| + 3|g'(\lambda)||g''(\lambda)| + 2|g'(\lambda)|^3 \leq \\ &\leq (\mathbb{E} \exp |X|) \left(\left(\frac{3}{e(1-|\lambda|)} \right)^3 + 3 \left(\frac{1}{e(1-|\lambda|)} \right) \left(\frac{2}{e(1-|\lambda|)} \right)^2 + 2 \left(\frac{1}{e(1-|\lambda|)} \right)^3 \right) \leq \\ &\leq \frac{3^3 + 3 \cdot 2^2 + 2}{e^3 (1-|\lambda|)^3} \mathbb{E} \exp |X| = \frac{41}{e^3} \frac{1}{(1-|\lambda|)^3} \mathbb{E} \exp |X|. \end{aligned}$$

Finally,

$$\begin{aligned} \left| f(\lambda) - \frac{1}{2} f''(0) \lambda^2 \right| &= \left| f(\lambda) - f(0) - f'(0) \lambda - \frac{1}{2} f''(0) \lambda^2 \right| \leq \\ &\leq \frac{1}{3!} |f'''(\theta \lambda)| |\lambda|^3 \leq \frac{41}{6e^3} \left(\frac{|\lambda|}{1-|\lambda|} \right)^3 \mathbb{E} \exp |X| \end{aligned}$$

for some $\theta \in [0, 1]$; and $\exp |X| \leq \exp(-X) + \exp X$. \square

We return to the functions $f_r(\cdot)$ introduced in (2a4) for a process X that satisfies Assumption 2a1.

2d2 Proposition. There exist $r_1, \varepsilon \in (0, \infty)$ such that $f_r(\varepsilon\lambda) \leq \lambda^2$ for all $r \in [r_1, 2r_1]$ and $\lambda \in [-1, 1]$.

Proof. Def. 1.4 ensures existence of r_1 such that $f_{2r_1}(\cdot)$ is bounded on some $[-\delta, \delta]$. Given $r \in [r_1, 2r_1]$, inequality 2a10(b) applied to r , $s = 2r_1 - r$ and $p = 2$ gives

$$2f_r\left(\frac{\lambda}{2}\sqrt{\frac{r}{2r_1}}\right) \leq f_{2r_1}(\lambda) + \frac{\lambda^2}{2r_1}$$

for $|\lambda| \leq \sqrt{2r_1}$. Thus, f_r is bounded on $[-\frac{\delta}{2\sqrt{2}}, \frac{\delta}{2\sqrt{2}}]$, uniformly on $r \in [r_1, 2r_1]$ (assuming $\delta \leq \sqrt{2r_1}$; otherwise use $\min(\delta, \sqrt{2r_1})$). And $\mathbb{E} \exp |\lambda S_r| \leq \exp f_r(-\lambda) + \exp f_r(\lambda)$ is bounded by some C for $|\lambda| \leq \frac{\delta}{2\sqrt{2}}$ and $r \in [r_1, 2r_1]$. Using the inequality $e^{\varepsilon x} - 1 \leq \varepsilon(e^x - 1)$ for $\varepsilon \in [0, 1]$ (and all x) we get $\mathbb{E} \exp \varepsilon |\lambda S_r| \leq 1 - \varepsilon + \varepsilon \mathbb{E} \exp |\lambda S_r| \leq 1 + (C - 1)\varepsilon$. We take ε such that $1 + (C - 1)\varepsilon \leq 2$ and get $\mathbb{E} \exp \frac{\varepsilon\delta}{2\sqrt{2}} |S_r| \leq 2$ for $r \in [r_1, 2r_1]$. By Lemma 2a8, $f_r(\frac{\varepsilon\delta}{2\sqrt{2}}\lambda) \leq \lambda^2$ for $|\lambda| \leq 1$. \square

2d3 Remark. (a) Using Prop. 2b5 we can serve all $r \in [r_1, \infty)$ by a single ε .

(b) On the other hand, $[r_1, 2r_1]$ may be replaced with $[\theta r_1, 2r_1]$ for arbitrary $\theta \in (0, 1]$ (but a small θ may require small ε).

(c) Combining (a) and (b) we can serve by a single ε all $r \in [c, \infty)$ for a given $c > 0$.

(d) In particular, for every r the function $f_r(\cdot)$ is finite on some neighborhood of 0 (but a small r may require small neighborhood).

2d4 Proposition. For every $r \in (0, \infty)$ there exists (evidently unique) $\sigma_r \in [0, \infty)$ such that for every $c \in (0, \infty)$ and every $\lambda \in (-c, c)$

$$\left| f_r(\lambda) - \frac{1}{2}\sigma_r^2\lambda^2 \right| \leq A \left(\frac{|\lambda|}{c - |\lambda|} \right)^3 (\exp f_r(-c) + \exp f_r(c));$$

here A is an absolute constant.

Proof. Nothing to prove when the right-hand side is infinite. When it is finite (which is ensured for small c by 2d3(d)) we apply Lemma 2d1 to the random variable cS_r and substitute λ/c for λ . (Of course, $\sigma_r^2 = \mathbb{E} S_r^2$, and $A = \frac{41}{6e^3} \approx 0.3402$ fits.) \square

3 The chain in action

3a Quadratic approximation

In this subsection we investigate an arbitrary family of functions $f_r : \mathbb{R} \rightarrow [0, \infty]$ for $r \in (0, \infty)$ that satisfy (2b1), (2c1) and Propositions 2d2, 2d4.

(These assumptions are satisfied by the functions introduced by (2a4) for a process X that satisfies Assumption 2a1.)

We denote $\text{Median}(a, b, c) = a + b + c - \min(a, b, c) - \max(a, b, c)$ for $a, b, c \in \mathbb{R}$.

3a1 Theorem. Let $\varepsilon \in (0, \sqrt{2} - 1)$, $r \in (0, \infty)$, and

$$f_r(\varepsilon\lambda\sqrt{r}) \leq \lambda^2 \quad \text{for } |\lambda| \leq 1.$$

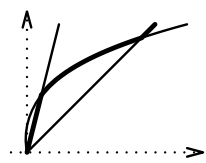
Then, for every $n = 2, 3, \dots$,

$$\left| \frac{1}{\lambda^2} f_{2^n r}(\varepsilon\lambda\sqrt{r}) - \frac{1}{2} r \sigma_{2^n r}^2 \varepsilon^2 \right| \leq A \cdot \text{Median}(|\lambda|, (2^{-n/2} n \varepsilon |\lambda|)^{1/3}, 2^{-n/2} |\lambda|)$$

for all λ such that $0 < |\lambda| \leq 2^{n/2} \min(\frac{1}{3n\varepsilon}, \frac{1}{9})$; here A is some absolute constant.

Remark 2b4 applies: Theorem 3a1 is scaling invariant.⁷

Note that

$$\begin{aligned} \text{Median}(|\lambda|, (2^{-n/2} n \varepsilon |\lambda|)^{1/3}, 2^{-n/2} |\lambda|) &= \\ &= \begin{cases} |\lambda| & \text{if } |\lambda| \leq \sqrt{n\varepsilon} \cdot 2^{-n/4}, \\ (2^{-n/2} n \varepsilon |\lambda|)^{1/3} & \text{if } \sqrt{n\varepsilon} \cdot 2^{-n/4} \leq |\lambda| \leq \sqrt{n\varepsilon} \cdot 2^{n/2}, \\ 2^{-n/2} |\lambda| & \text{if } \sqrt{n\varepsilon} \cdot 2^{n/2} \leq |\lambda|. \end{cases} \end{aligned}$$


3a2 Lemma. Let $\varepsilon \leq \sqrt{2} - 1$. If $f_1(\varepsilon\lambda) \leq \lambda^2$ for $|\lambda| \leq 1$, then

$$\left| f_{2^n}(\varepsilon\lambda) - \frac{1}{2} \sigma_{2^n}^2 \varepsilon^2 \lambda^2 \right| \leq A \left(\frac{|\lambda|}{1 - |\lambda|} \right)^3 \quad \text{for } |\lambda| < 1;$$

here A is an absolute constant.

Proof. By Prop. 2b5 for $a = 2$, $r = 1$ and $\lambda = \pm 1$,

$$f_{2^n}(\pm\varepsilon) \leq 2 \exp\left(\frac{\sqrt{2}}{\sqrt{2}-1}\varepsilon\right) - 1 \leq 2e^{\sqrt{2}} - 1.$$

By Prop. 2d4,

$$\left| f_{2^n}(\lambda) - \frac{1}{2} \sigma_{2^n}^2 \lambda^2 \right| \leq A_{2d4} \left(\frac{|\lambda|}{\varepsilon - |\lambda|} \right)^3 \cdot 2 \exp(2e^{\sqrt{2}} - 1) \quad \text{for } |\lambda| < \varepsilon,$$

that is,

$$\left| f_{2^n}(\varepsilon\lambda) - \frac{1}{2} \sigma_{2^n}^2 \varepsilon^2 \lambda^2 \right| \leq A \left(\frac{\varepsilon|\lambda|}{\varepsilon - \varepsilon|\lambda|} \right)^3 \quad \text{for } |\lambda| < 1.$$

where $A = 2 \exp(2e^{\sqrt{2}} - 1) A_{2d4}$. □

⁷Invariant are ε , λ , and $r\sigma^2$.

3a3 Lemma. $|\sigma_{2r} - \sigma_r| \leq \frac{1}{\sqrt{r}}$ for all $r \in (0, \infty)$.

Proof. Taking into account that $f_r(\lambda) = \frac{1}{2}\sigma_r^2\lambda^2 + o(\lambda^2)$, we get from (2b1)

$$\frac{1}{2}\sigma_{2r}^2 \leq p \cdot \frac{1}{2}\sigma_r^2 + \frac{p}{p-1} \cdot \frac{1}{2r} \quad \text{for all } p \in (1, \infty).$$

Taking $p = 1 + \frac{1}{\sigma_r\sqrt{r}}$ (the minimizer, in fact) we get $\frac{p}{p-1} = 1 + \sigma_r\sqrt{r}$, $\sigma_{2r}^2 \leq (\sigma_r + \frac{1}{\sqrt{r}})^2$, thus, $\sigma_{2r} \leq \sigma_r + \frac{1}{\sqrt{r}}$. It remains to prove that $\sigma_{2r} \geq \sigma_r - \frac{1}{\sqrt{r}}$.

By (2c1),

$$\frac{1}{2}\sigma_{2r}^2 \geq \frac{1}{p} \cdot \frac{1}{2}\sigma_r^2 - \frac{1}{p-1} \cdot \frac{1}{2r};$$

assuming $\sigma_r > \frac{1}{\sqrt{r}}$ (otherwise we have nothing to prove), we take $p = \frac{\sigma_r\sqrt{r}}{\sigma_r\sqrt{r}-1}$ (the minimizer), get $\frac{1}{p} = 1 - \frac{1}{\sigma_r\sqrt{r}}$, $\frac{1}{p-1} = \sigma_r\sqrt{r} - 1$, and $\sigma_{2r}^2 \geq (\sigma_r - \frac{1}{\sqrt{r}})^2$. \square

3a4 Corollary. $|\sigma_{2^n r} - \sigma_{2^m r}| \leq \frac{\sqrt{2}}{\sqrt{2}-1} \frac{1}{\sqrt{2^m r}}$ whenever $m \leq n$ and $r \in (0, \infty)$.

3a5 Proposition. Let $\varepsilon \in (0, \sqrt{2}-1)$, and

$$f_1(\varepsilon\lambda) \leq \lambda^2 \quad \text{for } |\lambda| \leq 1.$$

Then

$$\begin{aligned} \left| \frac{1}{\lambda^2} f_{2^n}(\varepsilon\lambda) - \frac{1}{2}\sigma_{2^n}^2 \varepsilon^2 \right| &\leq \frac{1}{2}\varepsilon^2 \left(\left(\sigma_{2^n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-n/2} \right)^2 - \sigma_{2^n}^2 \right) + \\ &\quad + A\delta + A\varepsilon(n2^{-n/2}|\lambda| + 2^{-n/2}\delta)\delta^{-2} \end{aligned}$$

for all $m \in \{0, 1, \dots, n-1\}$, $\delta \in (0, \frac{1}{2}]$ and λ such that

$$0 < |\lambda| \leq \frac{2^{n/2}\delta}{2n\varepsilon\delta + \max(\sqrt{2n\delta\varepsilon}, 2^{m/2})};$$

here A is some absolute constant.

Proof. Using Lemma 3a2 we have for $|\lambda| \leq \delta$

$$\left| f_{2^m}(\varepsilon\lambda) - \frac{1}{2}\sigma_{2^m}^2 \varepsilon^2 \lambda^2 \right| \leq A_{3a2} \left(\frac{|\lambda|}{1-|\lambda|} \right)^3 \leq A_{3a2} \frac{\delta}{(1-\delta)^3} \lambda^2 \leq 8A_{3a2} \delta \lambda^2,$$

thus,

$$(3a6) \quad f_{2^m}(\varepsilon\lambda) \leq \left(\frac{1}{2}\sigma_{2^m}^2 \varepsilon^2 + 8A_{3a2}\delta \right) \lambda^2 \quad \text{for } |\lambda| \leq \delta.$$

Now we need Theorem 2b8 rescaled as follows (a will be chosen later):

$$\varepsilon_{2b8} = 2^{-m/2}\delta\varepsilon, \quad r_{2b8} = 2^m, \quad a_{2b8} = 2^{-m}a, \quad n_{2b8} = n-m, \quad \lambda_{2b8} = 2^{m/2}\delta^{-1}\lambda.$$

We note that

$$(\varepsilon\lambda)_{2b8} = \varepsilon\lambda, \quad (a\lambda^2)_{2b8} = \delta^{-2}a\lambda^2, \quad (\lambda/\sqrt{r})_{2b8} = \delta^{-1}\lambda, \quad (2^n r)_{2b8} = 2^n.$$

The assumptions of 2b8 become: $2^{-m/2}\delta\varepsilon < \sqrt{2} - 1$ (holds evidently), and

$$f_{2m}(\varepsilon\lambda) \leq \delta^{-2}a\lambda^2 \quad \text{for } |\lambda| \leq \delta;$$

the latter holds by (3a6) provided that

$$(3a7) \quad a = \left(\frac{1}{2}\sigma_{2m}^2\varepsilon^2 + 8A_{3a2}\delta \right) \delta^2.$$

The conclusion of 2b8 becomes

$$(3a8) \quad f_{2n}(\varepsilon\lambda) \leq \left(a + C_{2b8} \cdot 2^{-m/2}\delta\varepsilon(a+1) \frac{1+V}{1-2^{-m/2}\delta\varepsilon V} \right) \delta^{-2}\lambda^2$$

for $|\lambda| \leq \frac{2^{n/2}\delta}{(n-m)\varepsilon\delta + \max(\sqrt{2(n-m)}\delta\varepsilon, 2^{m/2})}$ (which holds evidently),

where a is given by (3a7), $V = \frac{n-m}{2^{(n-m)/2}}\delta^{-1}|\lambda|$, and $C_{2b8} \leq \frac{2(e^{1/\sqrt{2}}-1)}{\sqrt{2}-1}$ (since $\varepsilon_{2b8} \leq \frac{\sqrt{2}-1}{2}$).

We have to prove two bounds, upper and lower, on $f_{2n}(\varepsilon\lambda)$. For the upper bound, it is sufficient to prove that

$$\begin{aligned} & \frac{1}{2}\sigma_{2m}^2\varepsilon^2 + 8A_{3a2}\delta + \left(C_{2b8} \cdot 2^{-m/2}\delta\varepsilon(a+1) \frac{1+V}{1-2^{-m/2}\delta\varepsilon V} \right) \delta^{-2} \leq \\ & \leq \frac{1}{2}\varepsilon^2 \left(\sigma_{2n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-m/2} \right)^2 + A\delta + A\varepsilon(n2^{-n/2}|\lambda| + 2^{-m/2}\delta) \delta^{-2}. \end{aligned}$$

By Corollary 3a4, $\sigma_{2m} \leq \sigma_{2n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-m/2}$; the needed upper bound inequality is reduced to

$$\begin{aligned} & 8A_{3a2}\delta + \left(\frac{2(e^{1/\sqrt{2}}-1)}{\sqrt{2}-1} \cdot 2^{-m/2}\delta\varepsilon(a+1) \frac{1+V}{1-2^{-m/2}\delta\varepsilon V} \right) \delta^{-2} \leq \\ & \leq A\delta + A\varepsilon(n2^{-n/2}|\lambda| + 2^{-m/2}\delta) \delta^{-2}, \end{aligned}$$

and further to⁸

$$(3a9) \quad 2^{-m/2}\delta(a+1) \frac{1+V}{1-2^{-m/2}\delta\varepsilon V} \leq A(n2^{-n/2}|\lambda| + 2^{-m/2}\delta).$$

⁸From now on, A denotes different absolute constants in different inequalities.

We note that $\sigma_1 \leq \sqrt{2}/\varepsilon$ (since $f_1(\varepsilon\lambda) \leq \lambda^2$); 3a4 gives $\varepsilon\sigma_{2^k} \leq \varepsilon\left(\frac{\sqrt{2}}{\varepsilon} + \frac{\sqrt{2}}{\sqrt{2}-1}\right) \leq 2\sqrt{2}$ for all k . By (3a7), $a \leq 1 + A_{3a2}$ (since $\delta \leq \frac{1}{2}$), which reduces 3a9 to

$$2^{-m/2}\delta \frac{1+V}{1-2^{-m/2}\delta\varepsilon V} \leq A(n2^{-n/2}|\lambda| + 2^{-m/2}\delta).$$

Further, $|\lambda| \leq \frac{2^{n/2}}{2n\varepsilon}$, thus

$$2^{-m/2}\delta\varepsilon V = 2^{-m/2}\delta\varepsilon \frac{n-m}{2^{(n-m)/2}}\delta^{-1}|\lambda| = \varepsilon(n-m)2^{-n/2}|\lambda| \leq \frac{1}{2},$$

which reduces 3a9 to

$$2^{-m/2}\delta(1+V) \leq A(n2^{-n/2}|\lambda| + 2^{-m/2}\delta),$$

and further, to $2^{-m/2}\delta V \leq An2^{-n/2}|\lambda|$, which holds (for $A = 1$) by the definition of V .

For the lower bound the proof is similar. First,

$$(3a10) \quad f_{2^m}(\varepsilon\lambda) \geq \left(\frac{1}{2}\sigma_{2^m}^2\varepsilon^2 - 8A_{3a2}\delta\right)\lambda^2 \quad \text{for } |\lambda| \leq \delta$$

similarly to (3a6). Second, the rescaling that was applied to Th. 2b8 applies now to Theorem 2c6. The assumptions of 2c6 become: $2^{-m/2}\delta\varepsilon < \sqrt{2}$ (holds evidently), and

$$f_{2^m}(\varepsilon\lambda) \geq \delta^{-2}a\lambda^2 \quad \text{for } |\lambda| \leq \delta;$$

the latter holds by (3a10) provided that

$$(3a11) \quad a = \left(\frac{1}{2}\sigma_{2^m}^2\varepsilon^2 - 8A_{3a2}\delta\right)\delta^2.$$

The conclusion of 2c6 becomes

$$(3a12) \quad f_{2^n}(\varepsilon\lambda) \geq (a - (\sqrt{2} + 1)2^{-m/2}\delta\varepsilon(a+1)(1+V))\delta^{-2}\lambda^2$$

for $2^{m/2}|\lambda| \leq 2^{n/2}\delta$ (which holds, since $|\lambda| \leq 2^{(n-m)/2}\delta$), where a is given by (3a11), and $V = \frac{n-m}{2^{(n-m)/2}}\delta^{-1}|\lambda|$ as before.

We replace $\left(\sigma_{2^n} + \frac{\sqrt{2}}{\sqrt{2}-1}2^{-m/2}\right)^2 - \sigma_{2^n}^2$ with $\sigma_{2^n}^2 - \left(\sigma_{2^n} - \frac{\sqrt{2}}{\sqrt{2}-1}2^{-m/2}\right)_+^2$ (the latter being smaller). It is sufficient to prove that

$$\begin{aligned} & \frac{1}{2}\sigma_{2^m}^2\varepsilon^2 - 8A_{3a2}\delta - (\sqrt{2} + 1)2^{-m/2}\delta\varepsilon(a+1)(1+V)\delta^{-2} \geq \\ & \geq \frac{1}{2}\varepsilon^2\left(\sigma_{2^n} - \frac{\sqrt{2}}{\sqrt{2}-1}2^{-m/2}\right)_+^2 - A\delta - A\varepsilon(n2^{-n/2}|\lambda| + 2^{-m/2}\delta)\delta^{-2}. \end{aligned}$$

By Corollary 3a4, $\sigma_{2^m} \geq \sigma_{2^n} - \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-m/2}$; the needed lower bound inequality is reduced to

$$\begin{aligned} 8A_{3a2}\delta + (\sqrt{2} + 1)2^{-m/2}\delta\varepsilon(a+1)(1+V)\delta^{-2} &\leq \\ &\leq A\delta + A\varepsilon(n2^{-n/2}|\lambda| + 2^{-m/2}\delta)\delta^{-2}, \end{aligned}$$

and further to

$$2^{-m/2}\delta(a+1)(1+V) \leq A(n2^{-n/2}|\lambda| + 2^{-m/2}\delta);$$

the latter holds by (3a9). \square

We start proving Theorem 3a1. According to Remark 2b4 we restrict ourselves to the case $r = 1$.

THE FIRST CASE: $|\lambda| \leq \sqrt{n\varepsilon} \cdot 2^{-n/4}$.

We note that $\sqrt{n\varepsilon} \cdot 2^{-n/4} \leq ((\sqrt{2} - 1) \max_n n 2^{-n/2})^{1/2} = ((\sqrt{2} - 1) \cdot 3 \cdot 2^{-3/2})^{1/2} < 2/3$ and apply Lemma 3a2: $|\frac{1}{\lambda^2} f_{2^n}(\varepsilon\lambda) - \frac{1}{2} \sigma_{2^n}^2 \varepsilon^2| \leq A_{3a2} \frac{|\lambda|}{(1-|\lambda|)^3} \leq 27A_{3a2}|\lambda|$.

THE SECOND CASE: $\sqrt{n\varepsilon} \cdot 2^{-n/4} \leq |\lambda| \leq 2^{n/2} \min(\sqrt{n\varepsilon}, \frac{1}{3n\varepsilon}, \frac{1}{9})$.

Before applying Prop. 3a5 we choose $m \in \{0, 1, \dots, n-1\}$ and $\delta \in (0, \frac{1}{2}]$ appropriately; namely, we want them to satisfy

$$(3a13) \quad \frac{\delta}{3} \leq (2^{-n/2} n \varepsilon |\lambda|)^{1/3} \leq 3\delta,$$

$$(3a14) \quad \frac{1}{\sqrt{2}}\delta \leq 3 \cdot 2^{-(n-m)/2} |\lambda| \leq \delta.$$

3a15 Lemma. Let $1 \leq a < \infty$, $0 < x < \infty$, $0 < \lambda \leq \min(\sqrt{x}, 1/x, 1/a^2)$. Then there exists $\delta \in (0, \infty)$ such that $a\delta \leq 1$, $a\lambda \leq \delta$, and

$$\frac{\delta}{a} \leq (x\lambda)^{1/3} \leq a\delta.$$

Proof. Existence of δ such that $\delta \leq a^{-1}$, $\delta \geq a\lambda$, $\delta \geq a^{-1}(x\lambda)^{1/3}$, $\delta \leq a(x\lambda)^{1/3}$ is equivalent to the inequality

$$\max(a\lambda, a^{-1}(x\lambda)^{1/3}) \leq \min(a^{-1}, a(x\lambda)^{1/3}),$$

thus, to the three inequalities

$$\begin{aligned} a\lambda &\leq a^{-1}, \quad \text{that is, } \lambda \leq a^{-2}; \\ a\lambda &\leq a(x\lambda)^{1/3}, \quad \text{that is, } \lambda \leq \sqrt{x}; \\ a^{-1}(x\lambda)^{1/3} &\leq a^{-1}, \quad \text{that is, } \lambda \leq x^{-1}. \end{aligned} \quad \square$$

Lemma 3a15, applied to $a = 3$, $x = n\varepsilon$ and $2^{-n/2}|\lambda|$ in place of λ , gives δ such that $3\delta \leq 1$ (and therefore $\delta < \frac{1}{2}$, as required), $3 \cdot 2^{-n/2}|\lambda| \leq \delta$, and (3a13) holds.

By (3a13), $(\frac{\delta}{3})^3 \leq 2^{-n/2}n\varepsilon|\lambda|$; on the other hand, $2^{-n/2}n\varepsilon \leq |\lambda|^2$; therefore $(\frac{\delta}{3})^3 \leq |\lambda|^3$, that is, $\delta \leq 3|\lambda|$.

Having $\frac{\delta}{3|\lambda|} \in [2^{-n/2}, 1] = \cup_{m=0}^{n-1} [2^{-(n-m)/2}, \sqrt{2} \cdot 2^{-(n-m)/2}]$, we take m such that $\frac{\delta}{3|\lambda|} \in [2^{-(n-m)/2}, \sqrt{2} \cdot 2^{-(n-m)/2}]$, which ensures (3a14).

In order to apply Prop. 3a5 we have to check that

$$(3a16) \quad |\lambda| \leq \frac{2^{n/2}\delta}{2n\varepsilon\delta + \max(\sqrt{2n\varepsilon\delta}, 2^{m/2})}.$$

We know that $|\lambda| \leq \frac{2^{n/2}}{3n\varepsilon}$; also, $|\lambda| \leq \frac{1}{3} \cdot 2^{(n-m)/2}\delta$ by (3a14); thus, (3a16) is reduced to

$$\min\left(\frac{1}{3n\varepsilon}, \frac{1}{3} \cdot 2^{-m/2}\delta\right) \cdot \max(2n\varepsilon\delta + \sqrt{2n\varepsilon\delta}, 2n\varepsilon\delta + 2^{m/2}) \leq \delta,$$

that is,

$$\min\left(\frac{1}{3n\varepsilon}, \frac{1}{3} \cdot 2^{-m/2}\delta\right) \cdot \max\left(2n\left(1 + \frac{1}{\sqrt{2n}}\right)\varepsilon, 2n\varepsilon + 2^{m/2}\delta^{-1}\right) \leq 1.$$

The left-hand side does not exceed⁹

$$\max\left(\frac{1}{3n\varepsilon}\left(1 + \frac{1}{2}\right) \cdot 2n\varepsilon, \frac{1}{3n\varepsilon} \cdot 2n\varepsilon + \frac{1}{3} \cdot 2^{-m/2}\delta \cdot 2^{m/2}\delta^{-1}\right) = \max(1, 1) = 1.$$

So, (3a16) holds; Prop. 3a5 applies, and gives the upper bound

$$\frac{1}{2}\varepsilon^2 \left(\left(\sigma_{2^n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-m/2} \right)^2 - \sigma_{2^n}^2 \right) + A_{3a5}\delta + A_{3a5}\varepsilon(n2^{-n/2}|\lambda| + 2^{-m/2}\delta)\delta^{-2};$$

we want to majorize this by $\text{const} \cdot (2^{-n/2}n\varepsilon|\lambda|)^{1/3}$ or, equivalently, by $\text{const} \cdot \delta$ (see (3a13)).

Below, $\mathcal{O}(x)$ means something majorized by $\text{const} \cdot x$ with some *absolute* constant. We have

$$(3a17) \quad \delta = \mathcal{O}(1) \quad \text{since } \delta \leq \frac{1}{2};$$

$$(3a18) \quad \varepsilon\sigma_{2^n} = \mathcal{O}(1) \quad \text{since } \varepsilon\sigma_{2^n} \leq 2\sqrt{2}, \text{ as noted after (3a9);}$$

$$(3a19) \quad 2^{-n/2}n\varepsilon|\lambda| = \mathcal{O}(\delta^3) \quad \text{by (3a13): } 2^{-n/2}n\varepsilon|\lambda| \leq (3\delta)^3;$$

$$(3a20) \quad 2^{-m/2} = \mathcal{O}(2^{-n/2}|\lambda|\delta^{-1}) \quad \text{by (3a14): } 2^{-m/2} \leq \sqrt{2} \cdot 3 \cdot 2^{-n/2}|\lambda|\delta^{-1};$$

$$(3a21) \quad 2^{-m/2}\varepsilon = \mathcal{O}(\delta^2) \quad \text{by (3a20) and (3a19).}$$

⁹Since $n \geq 2$, and $\min(x, y) \cdot \max(u, v + w) \leq \max(xu, xv + yw)$ for $u, v, w \geq 0$.

Thus, by (3a21) and (3a18),

$$\varepsilon^2 \left(\left(\sigma_{2^n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-m/2} \right)^2 - \sigma_{2^n}^2 \right) = \underbrace{\mathcal{O}(2^{-m/2} \varepsilon \cdot \varepsilon \sigma_{2^n})}_{=\mathcal{O}(\delta^2)} + \underbrace{\mathcal{O}(2^{-m} \varepsilon^2)}_{=\mathcal{O}(\delta^4)} = \mathcal{O}(\delta^2) = \mathcal{O}(\delta);$$

and finally, by (3a19) and (3a21),

$$\varepsilon(n 2^{-n/2} |\lambda| + 2^{-m/2} \delta) \delta^{-2} = (\mathcal{O}(\delta^3) + \mathcal{O}(\delta^2) \delta) \delta^{-2} = \mathcal{O}(\delta).$$

THE THIRD CASE: $2^{n/2} \sqrt{n\varepsilon} \leq |\lambda| \leq 2^{n/2} \min(\frac{1}{3n\varepsilon}, \frac{1}{9})$.

We want to apply Prop. 3a5 for $m = 0$ and $\delta = 3 \cdot 2^{-n/2} |\lambda|$ (as required, $\delta < \frac{1}{2}$). To this end we check that

$$|\lambda| \leq \frac{3|\lambda|}{6n\varepsilon \cdot 2^{-n/2} |\lambda| + \max(3\sqrt{2n} 2^{-n/2} |\lambda| \varepsilon, 1)},$$

that is,

$$\max((2n + \sqrt{2n}) \varepsilon 2^{-n/2} |\lambda|, 2n\varepsilon 2^{-n/2} |\lambda| + \frac{1}{3}) \leq 1.$$

The left-hand side does not exceed

$$\max\left(2n(1 + \frac{1}{2}) \varepsilon \cdot \frac{1}{3n\varepsilon}, 2n\varepsilon \cdot \frac{1}{3n\varepsilon} + \frac{1}{3}\right) = \max(1, 1) = 1.$$

So, Prop. 3a5 applies, and gives the upper bound

$$\begin{aligned} & \frac{1}{2} \varepsilon^2 \left(\left(\sigma_{2^n} + \frac{\sqrt{2}}{\sqrt{2}-1} 2^{-n/2} |\lambda| \right)^2 - \sigma_{2^n}^2 \right) + \\ & + 3A_{3a5} 2^{-n/2} |\lambda| + A_{3a5} \varepsilon (n 2^{-n/2} |\lambda| + 3 \cdot 2^{-n/2} |\lambda|) (3 \cdot 2^{-n/2} |\lambda|)^{-2} = \\ & = \mathcal{O}(\varepsilon^2 \sigma_{2^n}) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(2^{-n/2} |\lambda|) + \mathcal{O}\left(\frac{\varepsilon n 2^{n/2}}{|\lambda|}\right) + \mathcal{O}\left(\frac{2^{n/2} \varepsilon}{|\lambda|}\right); \end{aligned}$$

we want to majorize this by $2^{-n/2} |\lambda|$.

We have

(3a22)	$2^{-n/2} \lambda = \mathcal{O}(1)$	since $ \lambda \leq 2^{n/2} \cdot \frac{1}{9}$;
(3a23)	$\varepsilon \leq n\varepsilon = \mathcal{O}((2^{-n/2} \lambda)^2)$	since $2^{n/2} \sqrt{n\varepsilon} \leq \lambda $;
(3a24)	$\varepsilon = \mathcal{O}(2^{-n/2} \lambda)$	by (3a23) and (3a22);
	$\varepsilon^2 \sigma_{2^n} = \mathcal{O}(2^{-n/2} \lambda)$	by (3a24) and (3a18);
	$\varepsilon^2 = \mathcal{O}(2^{-n/2} \lambda)$	by (3a24) and (3a22);

and finally, $\frac{\varepsilon n 2^{n/2}}{|\lambda|} \leq \frac{(2^{-n/2} |\lambda|)^2 \cdot 2^{n/2}}{|\lambda|} = 2^{-n/2} |\lambda|$.

Theorem 3a1 is proved.

3a25 Corollary. Under the assumptions of Theorem 3a1,

$$\left| \frac{1}{\lambda^2} f_{2^{n_r}}(\varepsilon \lambda \sqrt{r}) - \frac{1}{2} r \sigma_{2^{n_r}}^2 \varepsilon^2 \right| \leq A(2^{-n/2} n \varepsilon |\lambda|)^{1/3}$$

for all λ such that $0 < |\lambda| \leq 2^{n/2} \min(\frac{1}{3n\varepsilon}, \frac{1}{9}, \sqrt{n\varepsilon})$.

3b Main result: proof

We return to the numbers σ_r introduced by Prop. 2d4. For every $r \in (0, \infty)$ the limit

$$\sigma_{2^\infty r} = \lim_{n \rightarrow \infty} \sigma_{2^{n_r}}$$

exists by 3a4.

3b1 Lemma. $\sigma_{2^\infty r}$ does not depend on r .

Proof. We'll prove that the function $r \mapsto r \sigma_{2^\infty r}^2$ is linear. It is sufficient to prove that it is additive,

$$(3b2) \quad (r+s) \sigma_{2^\infty(r+s)}^2 = r \sigma_{2^\infty r}^2 + s \sigma_{2^\infty s}^2,$$

and measurable.

For every λ the function $r \mapsto f_r(\lambda)$ is measurable due to (1.2), which implies measurability of the functions $r \mapsto \sigma_r^2 = \lim_{\lambda \rightarrow 0} \frac{2}{\lambda^2} f_r(\lambda)$ and $r \mapsto \sigma_{2^\infty r}$.

Multiplying by $\frac{2}{\lambda^2}$ the inequality 2a10(a) and taking the limit as $\lambda \rightarrow 0$ we get

$$\sigma_{r+s}^2 \leq \frac{1}{p} \cdot p^2 \frac{r}{r+s} \sigma_r^2 + \frac{1}{p} \cdot p^2 \frac{s}{r+s} \sigma_s^2 + \frac{p}{p-1} \frac{2}{r+s};$$

applying it to $2^n r$, $2^n s$ and taking the limit as $n \rightarrow \infty$ we get

$$\sigma_{2^\infty(r+s)}^2 \leq p \frac{r}{r+s} \sigma_{2^\infty r}^2 + p \frac{s}{r+s} \sigma_{2^\infty s}^2$$

for all $p > 1$ and therefore for $p = 1$. Similarly, the inequality

$$(r+s) \sigma_{2^\infty(r+s)}^2 \geq r \sigma_{2^\infty r}^2 + s \sigma_{2^\infty s}^2$$

follows from 2a10(b), and we get (3b2). □

Now we have $\sigma \in [0, \infty)$ such that $\sigma_{2^{n_r}} \rightarrow \sigma$ (as $n \rightarrow \infty$) for all $r \in (0, \infty)$; applying 3a4 to $m = 0$ and $n \rightarrow \infty$ we get

$$(3b3) \quad |\sigma_r - \sigma| \leq \frac{\sqrt{2}}{\sqrt{2}-1} \frac{1}{\sqrt{r}}; \quad \sigma_r \rightarrow \sigma \quad \text{as } r \rightarrow \infty.$$

Proof of Theorem 1.6. Assumption 2a1 applies without loss of generality. Remark 2d3(c) gives ε such that

$$f_r(\varepsilon\lambda) \leq \lambda^2 \quad \text{for all } \lambda \in [-1, 1] \text{ and } r \in [\tfrac{1}{2}, 1],$$

which ensures the condition of Th. 3a1: $f_r(\varepsilon\lambda\sqrt{r}) \leq \lambda^2$ for these λ and r (if $\varepsilon < \sqrt{2} - 1$; otherwise take a smaller ε). Corollary 3a25 applied to ε and $2^{-n}r$ gives, whenever $2^{-n}r \in [\tfrac{1}{2}, 1]$,

$$\left| \frac{1}{\lambda^2} f_r(\varepsilon\lambda 2^{-n/2}\sqrt{r}) - \frac{1}{2} \cdot 2^{-n}r\sigma_r^2\varepsilon^2 \right| \leq A(2^{-n/2}n\varepsilon|\lambda|)^{1/3}$$

for $0 < |\lambda| \leq 2^{n/2} \min(\frac{1}{3n\varepsilon}, \frac{1}{9}, \sqrt{n\varepsilon})$. We replace λ with $2^{n/2}\lambda/\varepsilon$:

$$\left| \frac{\varepsilon^2}{2^n\lambda^2} f_r(\lambda\sqrt{r}) - \frac{1}{2} \cdot 2^{-n}r\sigma_r^2\varepsilon^2 \right| \leq A(n|\lambda|)^{1/3}$$

for $0 < |\lambda| \leq \min(\frac{1}{3n}, \frac{\varepsilon}{9}, \varepsilon\sqrt{n\varepsilon})$. Thus,

$$\left| \frac{1}{r\lambda^2} f_r(\lambda\sqrt{r}) - \frac{1}{2}\sigma_r^2 \right| \leq A \cdot \frac{2^n}{\varepsilon^2 r} (n|\lambda|)^{1/3} \leq \frac{2A}{\varepsilon^2} (n|\lambda|)^{1/3} \leq \frac{2A}{\varepsilon^2} \left(|\lambda| \frac{\log 2r}{\log 2} \right)^{1/3}$$

is small whenever r is large and $|\lambda| \log r$ is small. Also, σ_r^2 is close to σ^2 by (3b3). \square

Proof of Corollary 1.7. Let $r_n \rightarrow \infty$, $c_n \rightarrow \infty$, $(c_n \log r_n)^2/r_n \rightarrow 0$; we have to prove that

$$\frac{1}{c_n^2} \log \mathbb{P} \left(\int_0^{r_n} X_t dt \geq c_n \sigma \sqrt{r_n} \right) \rightarrow -\frac{1}{2} \quad \text{as } n \rightarrow \infty.$$

Theorem 1.6 applied to r_n and $\lambda_n = \lambda c_n / \sqrt{r_n}$ gives

$$\frac{1}{c_n^2} \log \mathbb{E} \exp \frac{\lambda c_n}{\sqrt{r_n}} \int_0^{r_n} X_t dt \rightarrow \frac{\sigma^2}{2} \lambda^2 \quad \text{as } n \rightarrow \infty$$

for all $\lambda \in \mathbb{R}$. By the Gärtner-Ellis theorem [2] (with the scale c_n and speed c_n^2), random variables $\frac{1}{c_n \sqrt{r_n}} \int_0^{r_n} X_t dt$ satisfy MDP with the rate function $x \mapsto \frac{x^2}{2\sigma^2}$. \square

Proof of Corollary 1.8. For every $\lambda \neq 0$ Theorem 1.6 applied to r and λ/\sqrt{r} gives

$$\frac{1}{r(\lambda/\sqrt{r})^2} \log \mathbb{E} \exp \frac{\lambda}{\sqrt{r}} \int_0^r X_t dt \rightarrow \frac{\sigma^2}{2} \quad \text{as } r \rightarrow \infty,$$

that is,

$$\mathbb{E} \exp \lambda \cdot \frac{1}{\sqrt{r}} \int_0^r X_t dt \rightarrow \exp \left(\frac{1}{2} \sigma^2 \lambda^2 \right) \quad \text{as } t \rightarrow \infty.$$

The weak convergence of distributions follows, see for example [1, Sect. 30, p. 390]. \square

References

- [1] P. Billingsley (1995): *Probability and measure* (third edition), Wiley.
- [2] R.S. Ellis (2006): *The theory of large deviations and applications to statistical mechanics*, <http://www.math.umass.edu/~rsellis/pdf-files/Dresden-lectures.pdf>

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